Reflection and Transmission at a Potential Step

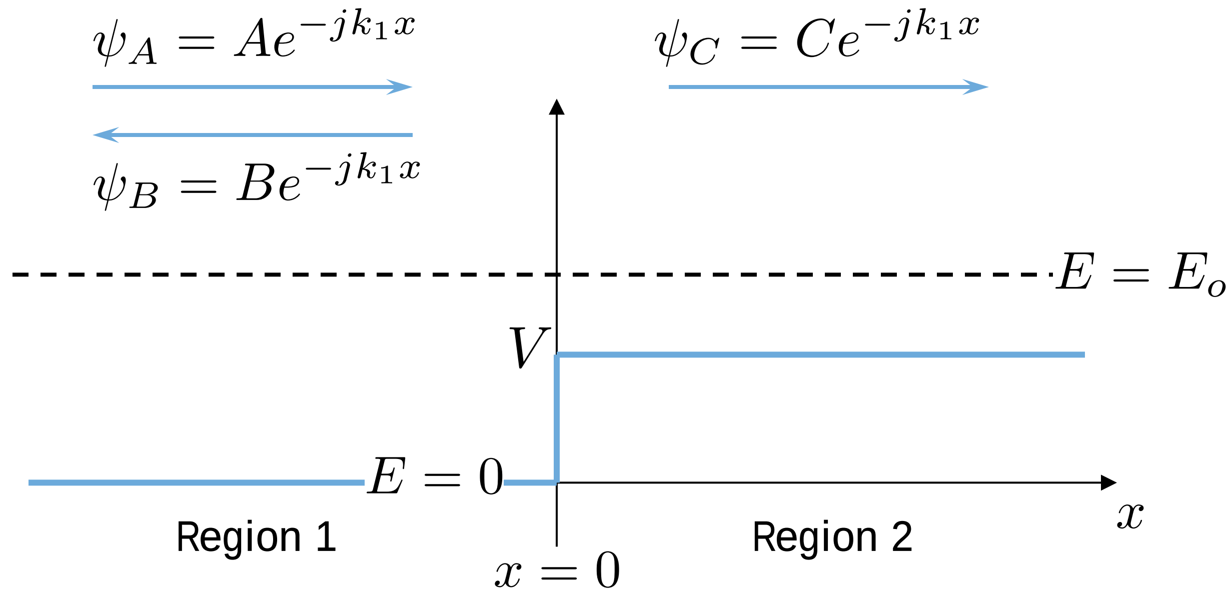
Outline

- Review: Particle in a 1-D Box
- Reflection and Transmission - Potential Step
- Reflection from a Potential Barrier
- Introduction to Barrier Penetration (Tunneling)

Reading and Applets:

.Text on Quantum Mechanics by French and Taylor
.Tutorial 10 – Quantum Mechanics in 1-D Potentials
.applets at http://phet.colorado.edu/en/get-phet/one-at-a-time

A Simple Potential Step



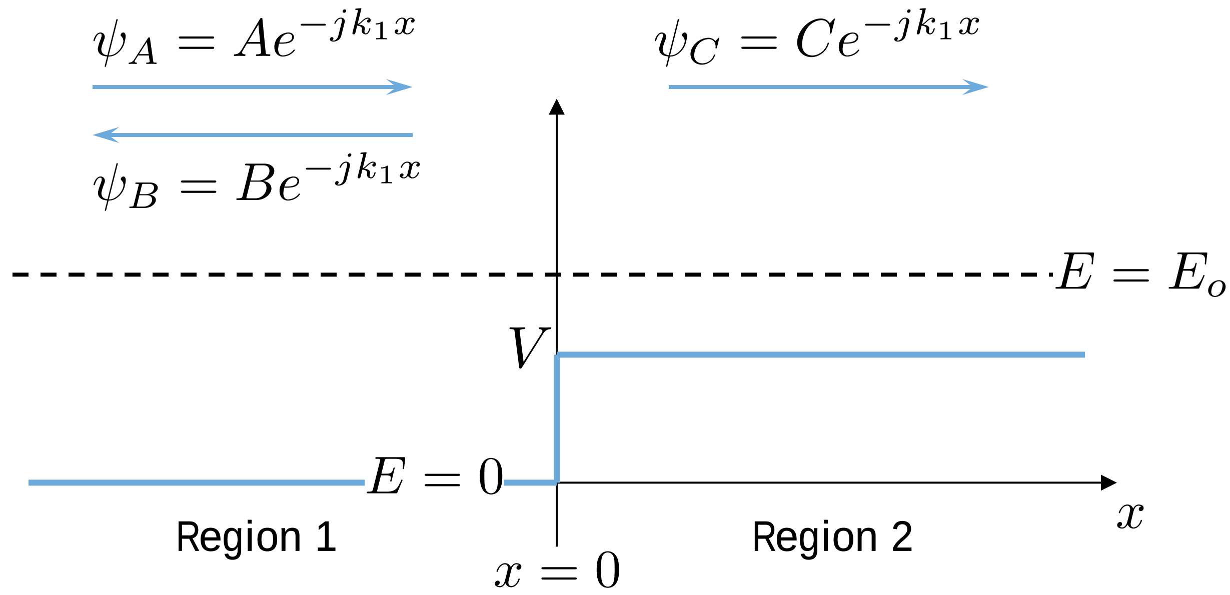
CASE I : $E_o > V$

In Region 1: $$E_o \psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \quad \Rightarrow \quad k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2: $$\left(E_o - V\right)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \quad \Rightarrow \quad k_2^2 = \frac{2m\left(E_o - V\right)}{\hbar^2}$$

A Simple Potential Step

CASE I : $E_o > V$



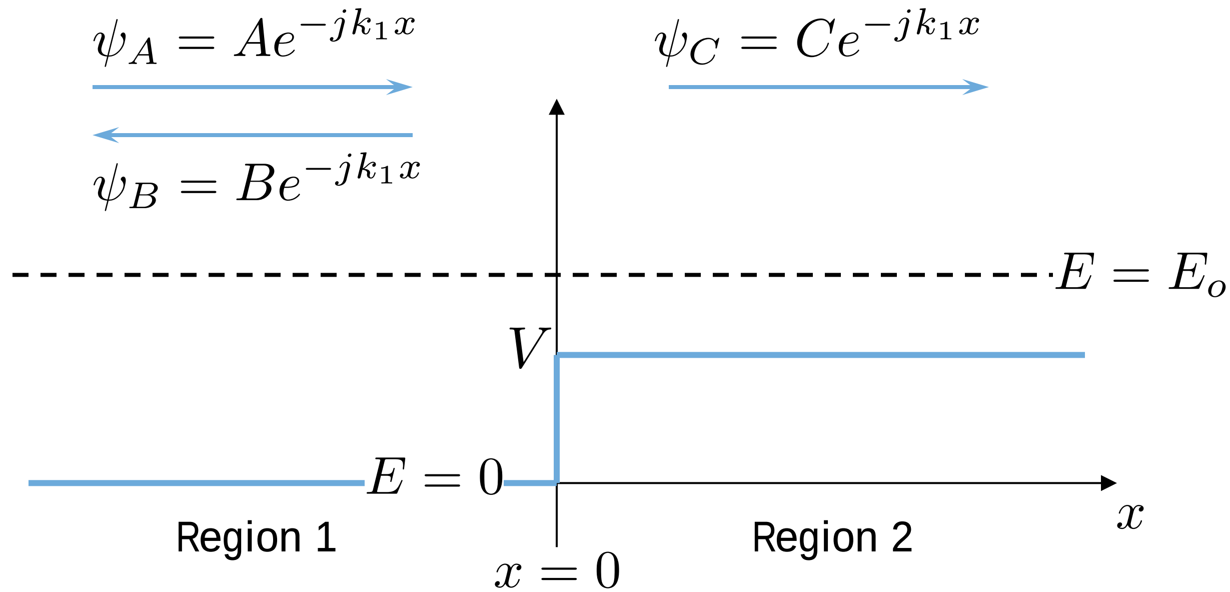
$$\psi_1 = Ae^{-jk_1x} + Be^{jk_1x} \qquad \psi_2 = Ce^{-jk_2x}$$

ψ is continuous: $\psi_1(0) = \psi_2(0)$ ⇒ $A + B = C$

$\frac{\partial \psi}{\partial x}$ is continuous: $\frac{\partial}{\partial x}\psi(0) = \frac{\partial}{\partial x}\psi_2(0)$ ⇒ $A - B = \frac{k_2}{k_1}C$

A Simple Potential Step

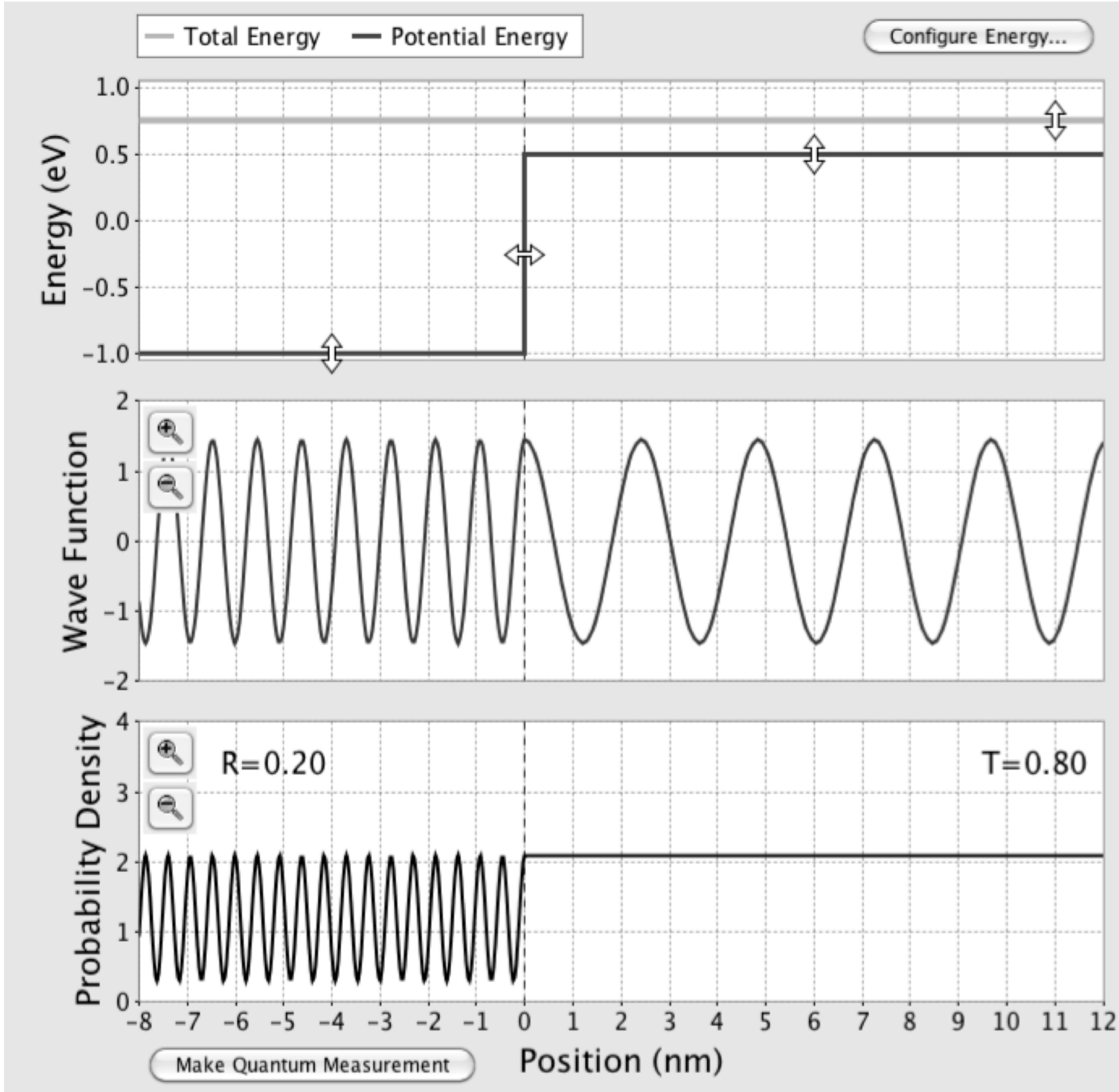
CASE I : $E_o > V$



$$\frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} = \frac{k_1 - k_2}{k_1 + k_2}$$

$$\frac{C}{A} = \frac{2}{1 + k_2/k_1} = \frac{2k_1}{k_1 + k_2}$$

$$\begin{cases} A + B = C \\ A - B = \dfrac{k_2}{k_1} C \end{cases}$$



Example from: [*http://phet.colorado.edu/en/get-phet/one-at-a-time*](http://phet.colorado.edu/en/get-phet/one-at-a-time)

Quantum Electron Currents

Given an electron of mass m

that is located in space with charge density $\rho = q\left|\psi(x)\right|^2$

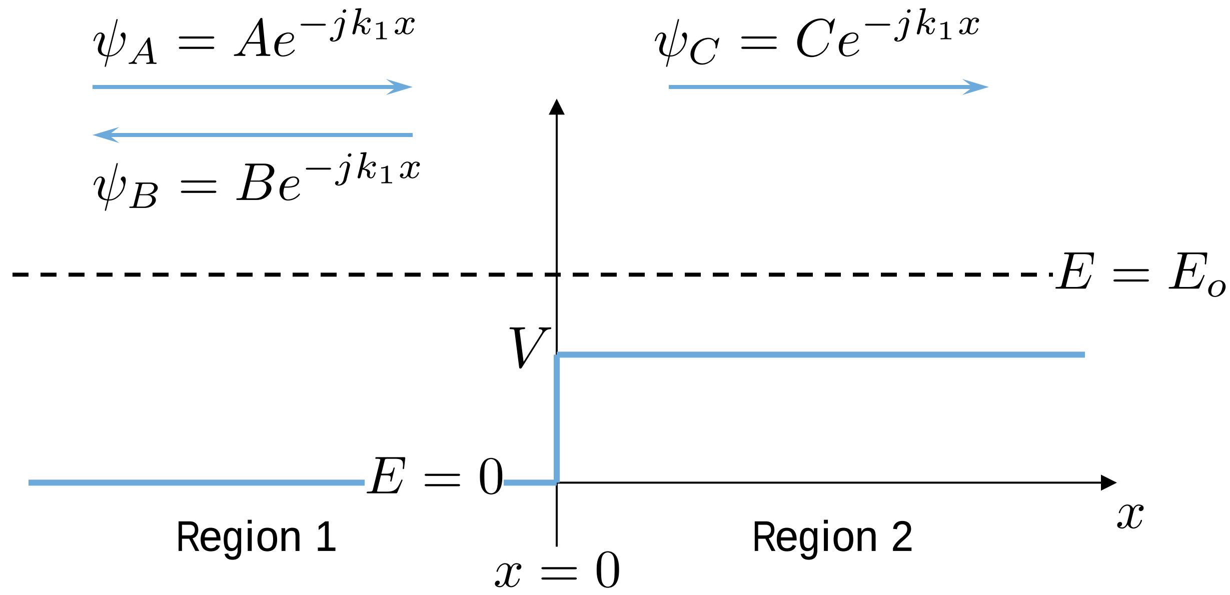
and moving with momentum $<p>$ corresponding to $<v> = \hbar k/m$

... then the current density for a *single electron* is given by

$$\boxed{J = \rho v = q\left|\psi\right|^2 (\hbar k/m)}$$

A Simple Potential Step

CASE I : $E_o > V$

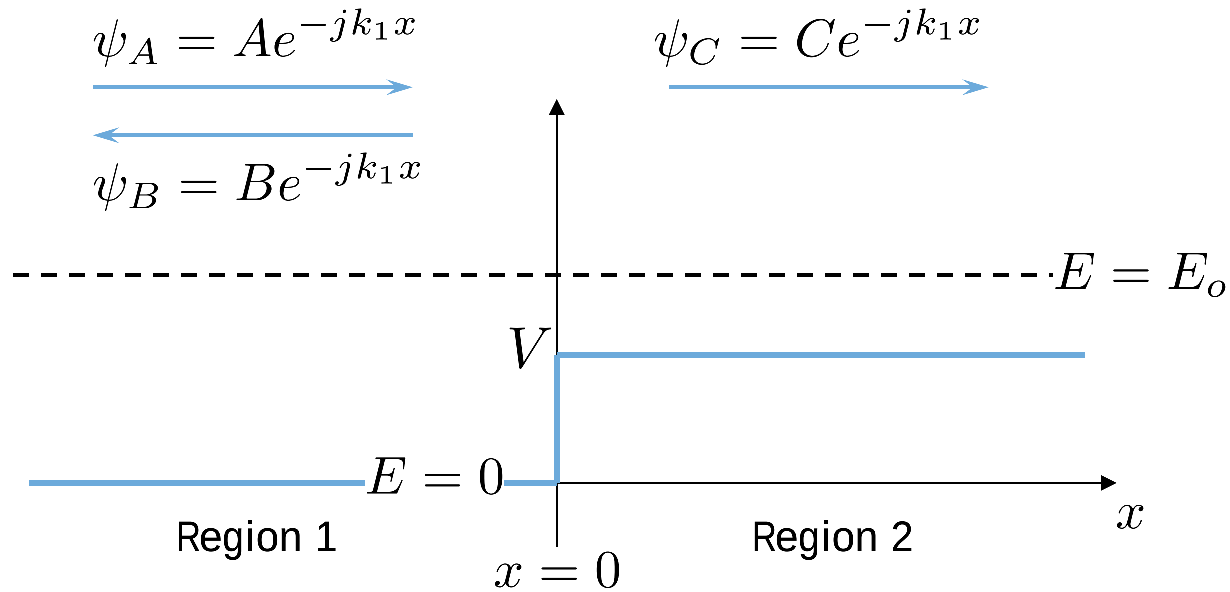


$$\text{Reflection} = R = \frac{J_{reflected}}{J_{incident}} = \frac{J_B}{J_A} = \frac{|\psi_B|^2(\hbar k_1/m)}{|\psi_A|^2(\hbar k_1/m)} = \left|\frac{B}{A}\right|^2$$

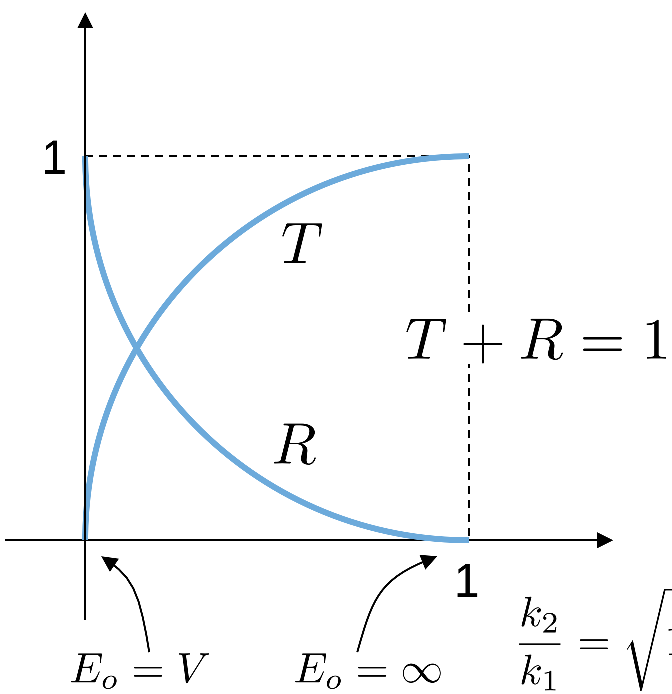
$$\text{Transmission} = T = \frac{J_{transmitted}}{J_{incident}} = \frac{J_C}{J_A} = \frac{|\psi_C|^2(\hbar k_2/m)}{|\psi_A|^2(\hbar k_1/m)} = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_1}$$

$$\frac{B}{A} = \frac{1 - k_2/k_1}{1 + k_2/k_1} \qquad \frac{C}{A} = \frac{2}{1 + k_2/k_1}$$

A Simple Potential Step



CASE I : E_o > V



$$\frac{k_2}{k_1} = \sqrt{1 - \frac{V}{E_o}}$$

$$\text{Reflection} = R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

$$\text{Transmission} = T = 1 - R$$

$$= \frac{4k_1 k_2}{|k_1 + k_2|^2}$$

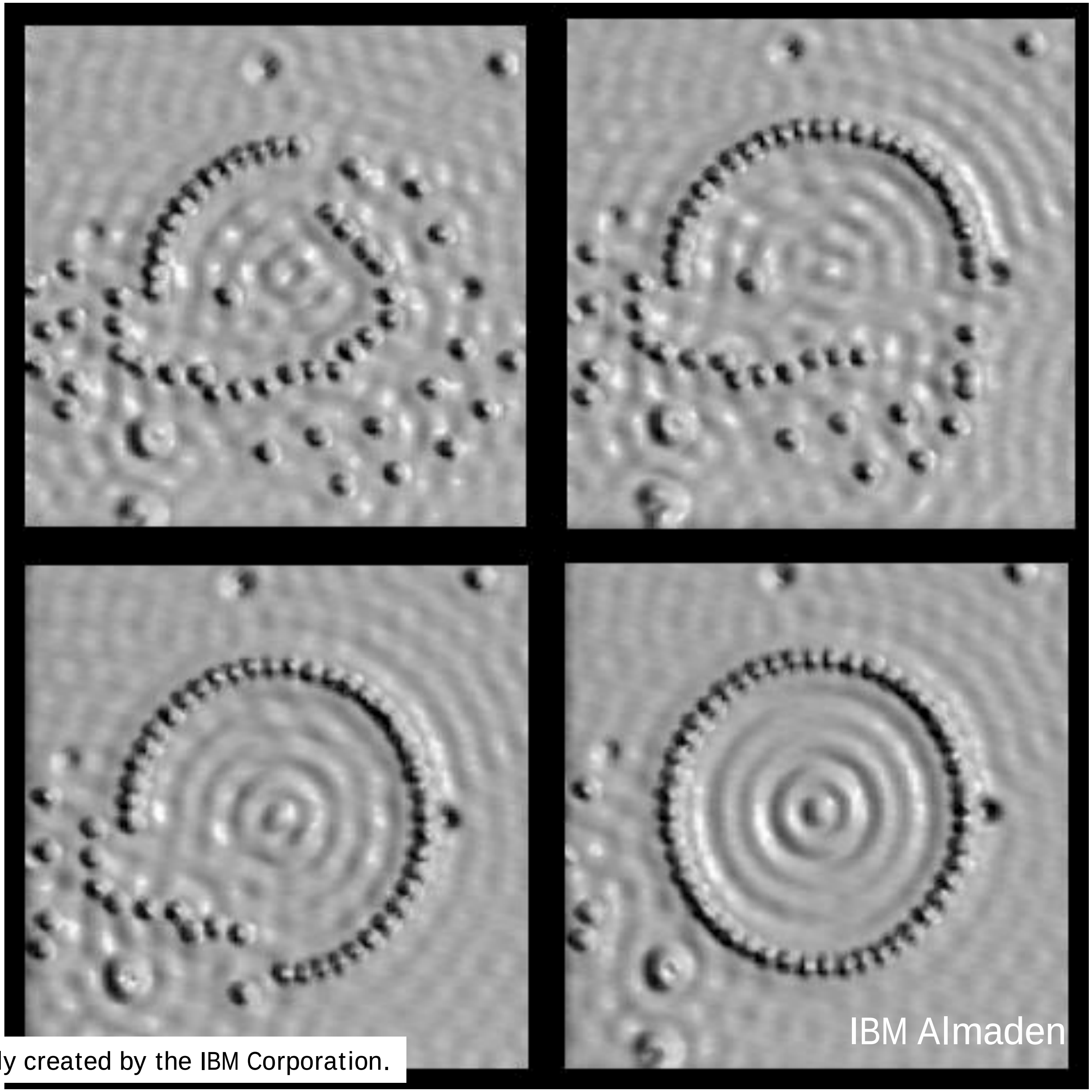


Image originally created by the IBM Corporation.

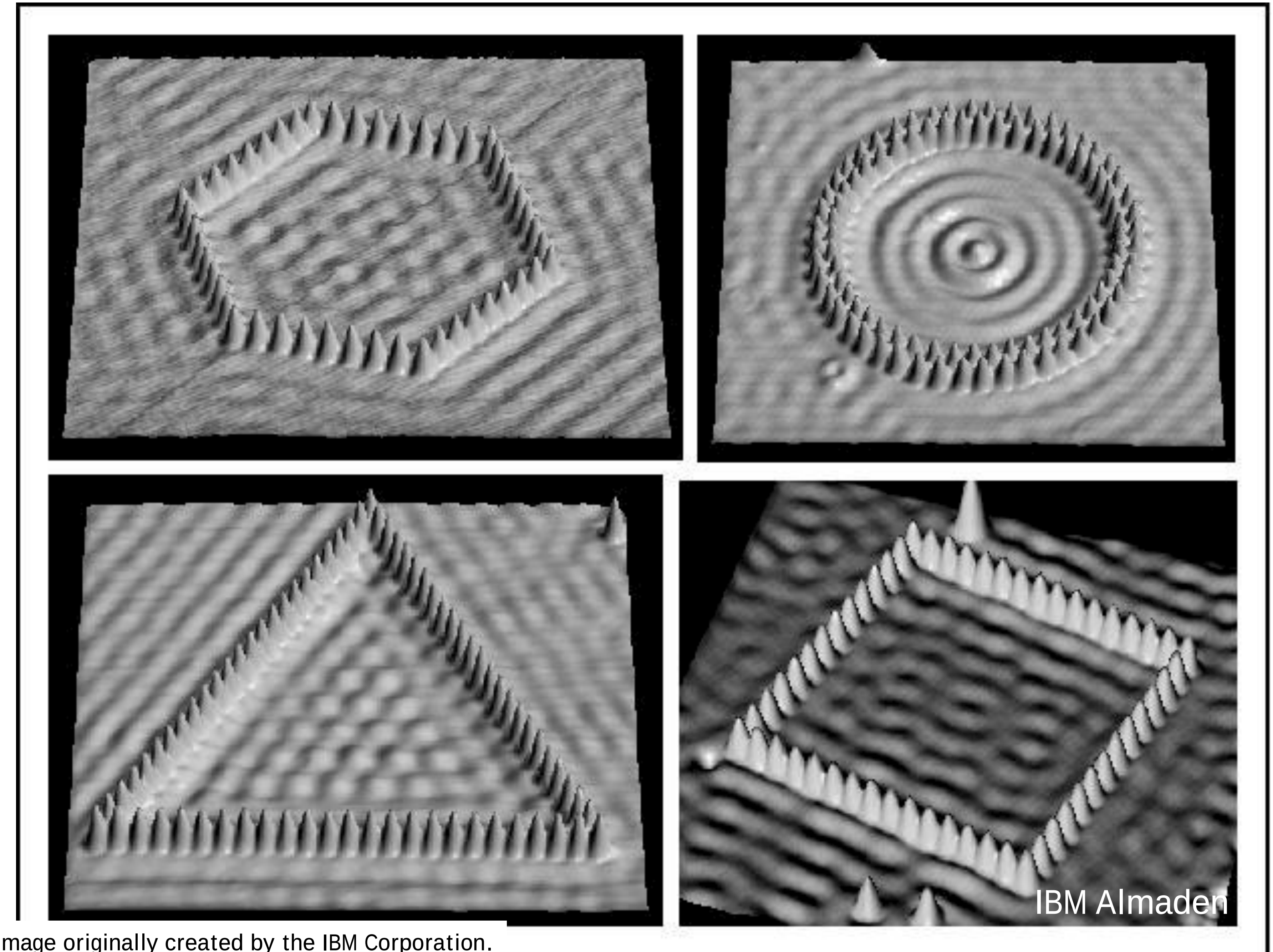
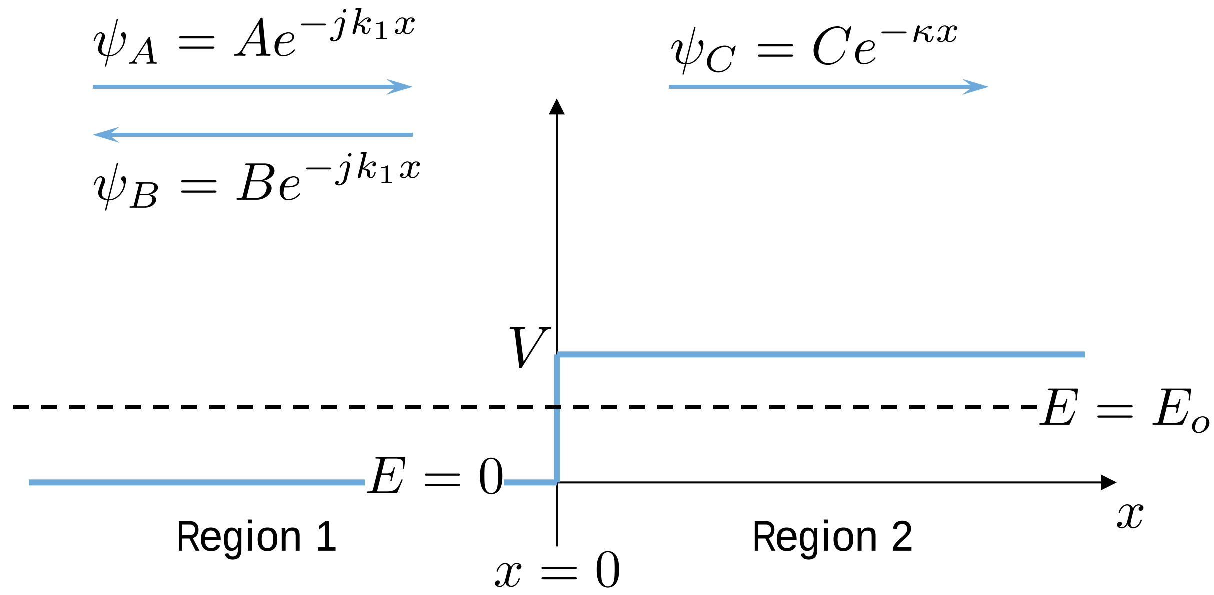


Image originally created by the IBM Corporation.

A Simple Potential Step

CASE II : $E_o < V$

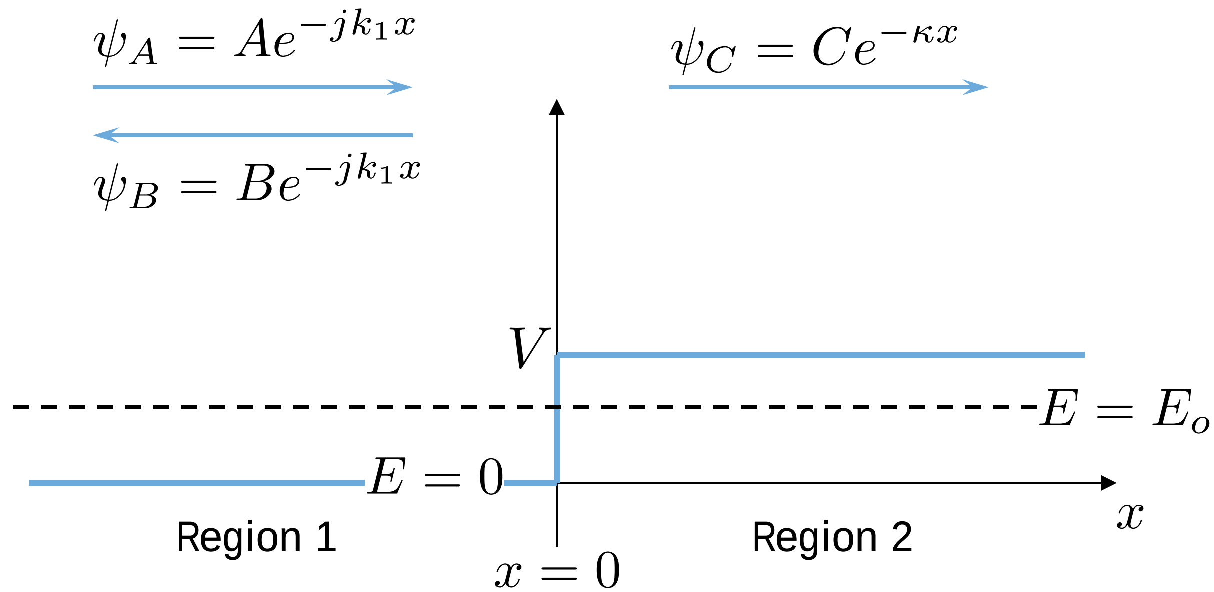


In Region 1:
$$E_o\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2:
$$\left(E_o - V\right)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow \kappa^2 = \frac{2m\left(E_o - V\right)}{\hbar^2}$$

A Simple Potential Step

CASE II : $E_o < V$



$$\psi_1 = Ae^{-jk_1x} + Be^{jk_1x} \qquad \psi_2 = Ce^{-\kappa x}$$

ψ is continuous: $\psi_1(0) = \psi_2(0) \Rightarrow A + B = C$

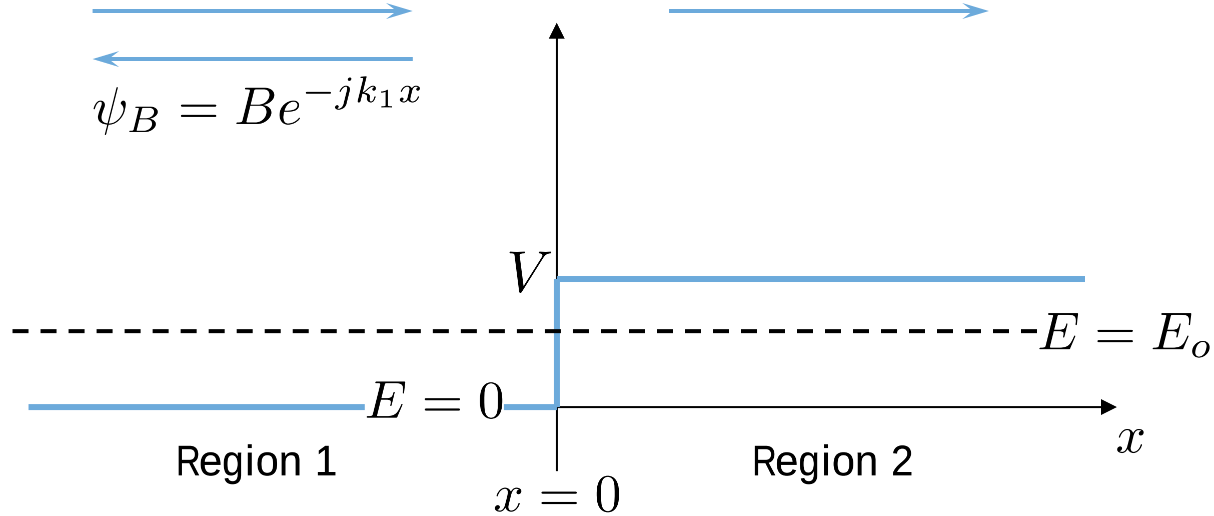
$\dfrac{\partial \psi}{\partial x}$ is continuous: $\dfrac{\partial}{\partial x}\psi(0) = \dfrac{\partial}{\partial x}\psi_2(0) \Rightarrow A - B = -j\dfrac{\kappa}{k_1}C$

A Simple Potential Step

$$\psi_A = Ae^{-jk_1x} \qquad \psi_C = Ce^{-\kappa x}$$

$$\psi_B = Be^{-jk_1x}$$

CASE II : $E_o < V$



$$\begin{cases} A + B = C \\ A - B = -j\dfrac{\kappa}{k_1}C \end{cases}$$

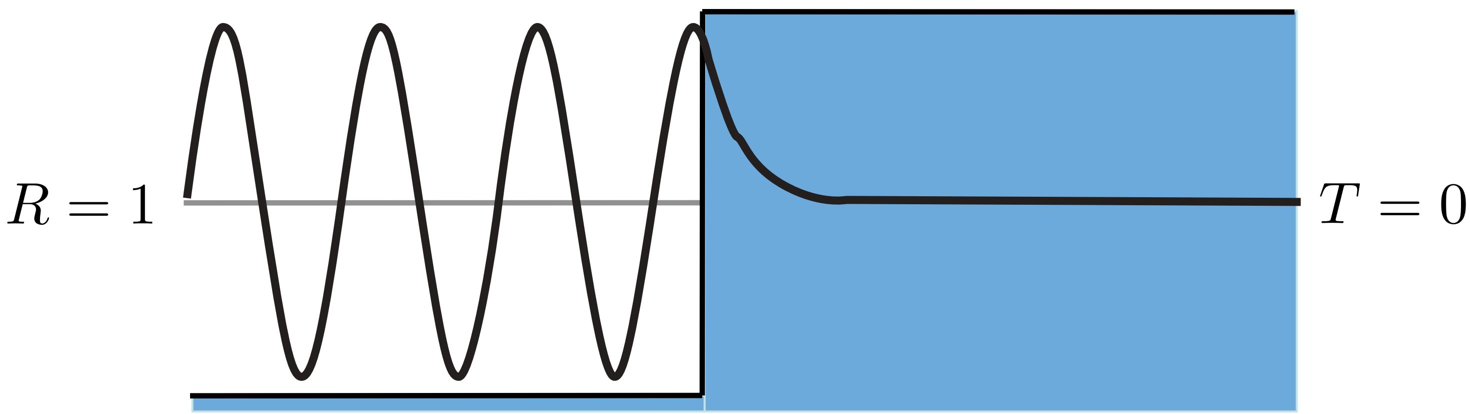
$$\frac{B}{A} = \frac{1 + j\kappa/k_1}{1 - j\kappa/k_1} \qquad \frac{C}{A} = \frac{2}{1 - j\kappa/k_1}$$

$$\boxed{R = \left|\frac{B}{A}\right|^2 = 1 \qquad T = 0}$$

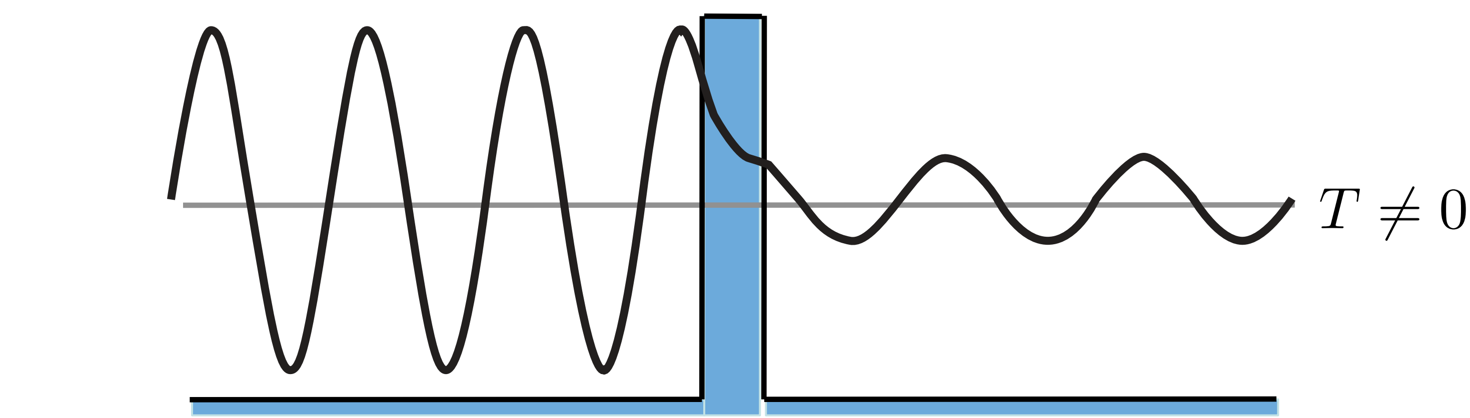
Total reflection → Transmission must be zero

Quantum Tunneling Through a Thin Potential Barrier

Total Reflection at Boundary



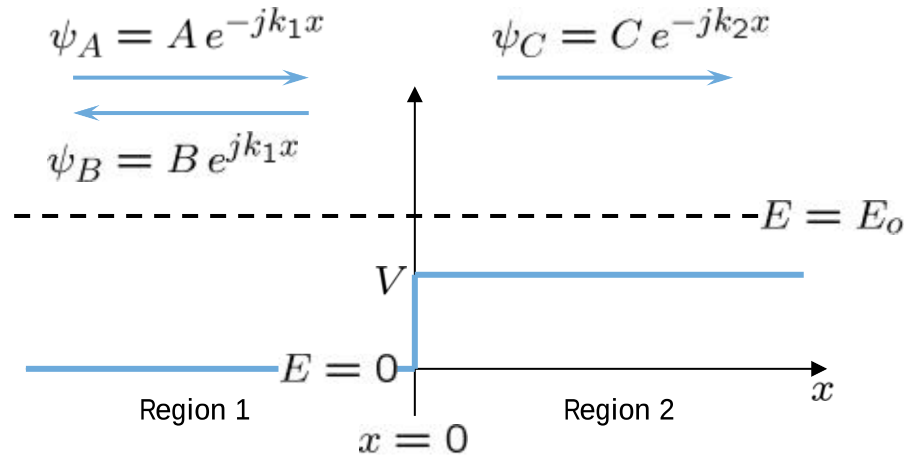
Frustrated Total Reflection (Tunneling)



KEY TAKEAWAYS

A Simple Potential Step

CASE I : $E_o > V$



$$Reflection = R = \left|\frac{B}{A}\right|^2 = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2$$

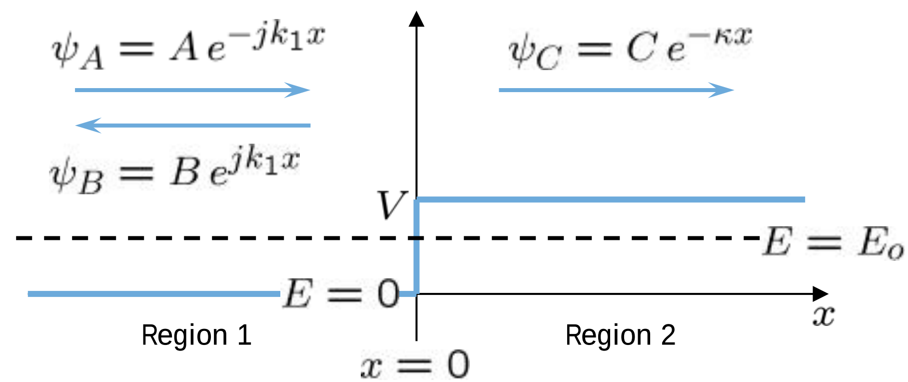
$$Transmission = T = 1 - R = \frac{4\, k_1 k_2}{|k_1 + k_2|^2}$$

$$\psi_1 = A\, e^{-jk_1 x} + B\, e^{jk_1 x} \qquad \psi_2 = C\, e^{-jk_2 x}$$

$$k_1^2 = \frac{2m\, E_o}{\hbar^2} \qquad k_2^2 = \frac{2m\,(E_o - V)}{\hbar^2}$$

PARTIAL REFLECTION

CASE II : $E_o < V$



$$R = |\frac{B}{A}|^2 = 1$$

$$T = 0$$

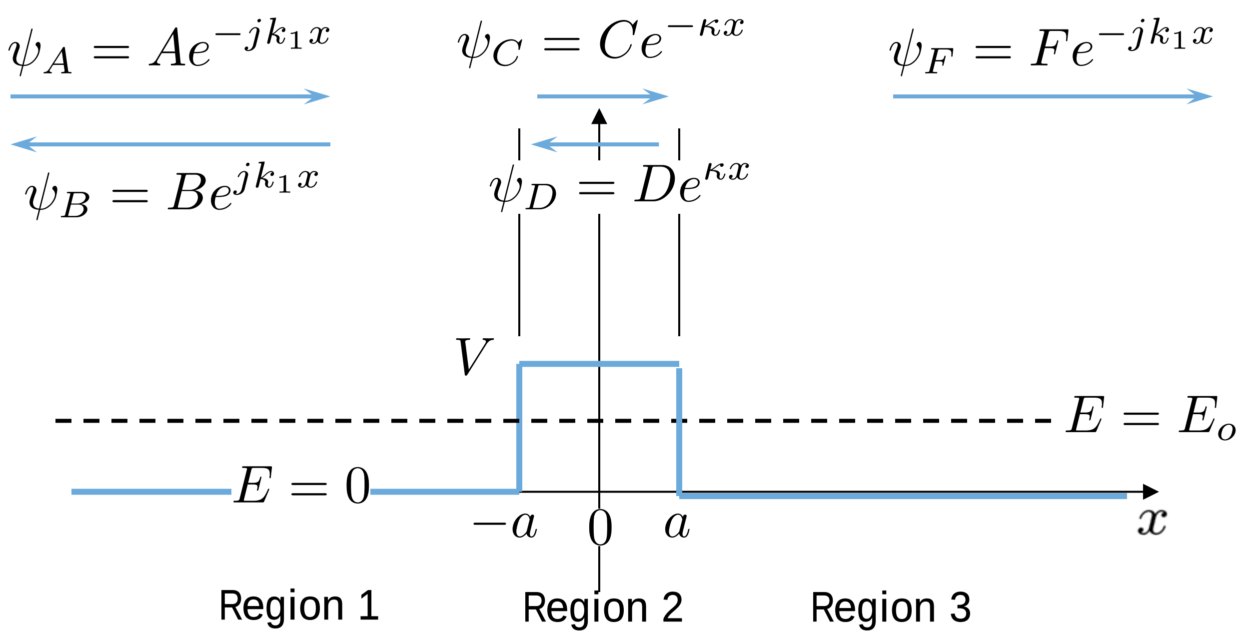
$$\psi_1 = A\, e^{-jk_1 x} + B\, e^{jk_1 x} \qquad \psi_2 = C\, e^{-\kappa x}$$

$$k_1^2 = \frac{2m\, E_o}{\hbar^2} \qquad \kappa^2 = \frac{2m\,(V - E_o)}{\hbar^2}$$

TOTAL REFLECTION

A Rectangular Potential Step

CASE II : $E_o < V$



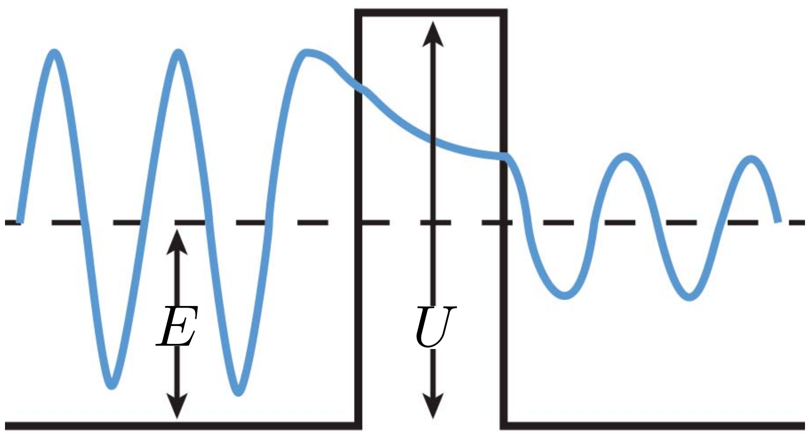
In Regions 1 and 3: $$E_o\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow k_1^2 = \frac{2mE_o}{\hbar^2}$$

In Region 2: $$(E_o - V)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \Rightarrow \kappa^2 = \frac{2m(V - E_o)}{\hbar^2}$$

for $E_o < V$:

$$\boxed{T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{1}{4}\frac{V^2}{E_o(V-E_o)}\sinh^2(2\kappa a)}}$$

A Rectangular Potential Step



for $E_o < V$:

$$\boxed{T = \left|\frac{F}{A}\right|^2 = \frac{1}{1 + \frac{1}{4}\frac{V^2}{E_o(V-E_o)}\sinh^2(2\kappa a)}}$$

$$\sinh^2(2\kappa a) = \left[e^{2\kappa a} - e^{-2\kappa a}\right]^2 \approx e^{-4\kappa a}$$

$$\boxed{T = \left|\frac{F}{A}\right|^2 \approx \frac{1}{1 + \frac{1}{4}\frac{V^2}{E_o(V-E_o)}} e^{-4\kappa a}}$$